9.4 Solve Quadratic Equations in Quadratic Form

Learning Objectives

By the end of this section, you will be able to:

- Solve equations in quadratic form

Be Prepared!

Before you get started, take this readiness quiz.

1. Factor by substitution: $y^4 - y^2 - 20$.
   If you missed this problem, review Example 6.21.
2. Factor by substitution: $(y-4)^2 + 8(y-4) + 15$.
   If you missed this problem, review Example 6.22.
3. Simplify: ⓐ $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$ ⓑ $\left(x^{\frac{1}{3}}\right)^2$ ⓒ $\left(x^{-1}\right)^2$.
   If you missed this problem, review Example 8.33.

Solve Equations in Quadratic Form

Sometimes when we factored trinomials, the trinomial did not appear to be in the $ax^2 + bx + c$ form. So we factored by substitution allowing us to make it fit the $ax^2 + bx + c$ form. We used the standard u for the substitution.

To factor the expression $x^4 - 4x^2 - 5$, we noticed the variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $\left(x^2\right)^2 = x^4$.) So we let $u = x^2$ and factored.

	$x^4 - 4x^2 - 5$
	$(x^2)^2 - 4(x^2) - 5$
Let $u = x^2$ and substitute.	$u^2 - 4u - 5$
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$

Similarly, sometimes an equation is not in the $ax^2 + bx + c = 0$ form but looks much like a quadratic equation. Then, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c = 0$ form. If we can make it fit the form, we can then use all of our methods to solve quadratic equations.

Notice that in the quadratic equation $ax^2 + bx + c = 0$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Again, we will use the standard u to make a substitution that will put the equation in quadratic form. If the substitution gives us an equation of the form $ax^2 + bx + c = 0$, we say the original equation was of **quadratic form**.

The next example shows the steps for solving an equation in quadratic form.

EXAMPLE 9.30 HOW TO SOLVE EQUATIONS IN QUADRATIC FORM

Solve: $6x^4 - 7x^2 + 2 = 0$

Solution

Step 1. Identify a substitution that will put the equation in quadratic form.	Since $(x^2)^2 = x^4$, we let $u = x^2$.	$6x^4 - 7x^2 + 2 = 0$
Step 2. Rewrite the equation with the substitution to put it in quadratic form.	Rewrite to prepare for the substitution. Substitute $u = x^2$.	$6(x^2)^2 - 7x^2 + 2 = 0$ $6u^2 - 7u + 2 = 0$
Step 3. Solve the quadratic equation for u.	We can solve by factoring. Use the Zero Product Property.	$(2u - 1)(3u - 2) = 0$ $2u - 1 = 0,\ 3u - 2 = 0$ $2u = 1,\ 3u = 2$ $u = \frac{1}{2} \quad u = \frac{2}{3}$
Step 4. Substitute the original variable back into the results, using the substitution.	Replace u with x^2.	$x^2 = \frac{1}{2} \qquad x^2 = \frac{2}{3}$
Step 5. Solve for the original variable.	Solve for x, using the Square Root Property.	$x = \pm\sqrt{\frac{1}{2}} \qquad x = \pm\sqrt{\frac{2}{3}}$ $x = \pm\frac{\sqrt{2}}{2} \qquad x = \pm\frac{\sqrt{6}}{3}$ There are four solutions. $x = \frac{\sqrt{2}}{2} \qquad x = \frac{\sqrt{6}}{3}$ $x = -\frac{\sqrt{2}}{2} \qquad x = -\frac{\sqrt{6}}{3}$
Step 6. Check the solutions.	Check all four solutions. We will show one check here.	$x = \frac{\sqrt{2}}{2}$ $6x^4 - 7x^2 + 2 = 0$ $6\left(\frac{\sqrt{2}}{2}\right)^4 - 7\left(\frac{\sqrt{2}}{2}\right)^2 + 2 \stackrel{?}{=} 0$ $6\left(\frac{4}{16}\right) - 7\left(\frac{2}{4}\right)^2 + 2 \stackrel{?}{=} 0$ $\frac{3}{2} - \frac{7}{2} + \frac{4}{2} \stackrel{?}{=} 0$ $0 = 0$ ✓ We leave the other checks to you!

TRY IT : : 9.59 Solve: $x^4 - 6x^2 + 8 = 0$.

TRY IT : : 9.60 Solve: $x^4 - 11x^2 + 28 = 0$.

We summarize the steps to solve an equation in quadratic form.

HOW TO : : SOLVE EQUATIONS IN QUADRATIC FORM.

Step 1. Identify a substitution that will put the equation in quadratic form.
Step 2. Rewrite the equation with the substitution to put it in quadratic form.
Step 3. Solve the quadratic equation for u.
Step 4. Substitute the original variable back into the results, using the substitution.
Step 5. Solve for the original variable.
Step 6. Check the solutions.

In the next example, the binomial in the middle term, $(x - 2)$ is squared in the first term. If we let $u = x - 2$ and substitute, our trinomial will be in $ax^2 + bx + c$ form.

EXAMPLE 9.31

Solve: $(x-2)^2 + 7(x-2) + 12 = 0$.

Solution

	$(x-2)^2 + 7(x-2) + 12 = 0$
Prepare for the substitution.	$(x-2)^2 + 7(x-2) + 12 = 0$
Let $u = x - 2$ and substitute.	$u^2 + 7u + 12 = 0$
Solve by factoring.	$(u+3)(u+4) = 0$ $u+3=0,\quad u+4=0$ $u=-3,\quad u=-4$
Replace u with $x-2$.	$x-2=-3,\quad x-2=-4$
Solve for x.	$x=-1,\quad x=-2$

Check:

$x=-1$	$x=-2$
$(x-2)^2 + 7(x-2) + 12 = 0$	$(x-2)^2 + 7(x-2) + 12 = 0$
$(-1-2)^2 + 7(-1-2) + 12 \overset{?}{=} 0$	$(-2-2)^2 + 7(-2-2) + 12 \overset{?}{=} 0$
$(-3)^2 + 7(-3) + 12 \overset{?}{=} 0$	$(-4)^2 + 7(-4) + 12 \overset{?}{=} 0$
$9 - 21 + 12 \overset{?}{=} 0$	$16 - 28 + 12 \overset{?}{=} 0$
$0 = 0$ ✓	$0 = 0$ ✓

TRY IT : : 9.61 Solve: $(x-5)^2 + 6(x-5) + 8 = 0$.

TRY IT : : 9.62 Solve: $(y-4)^2 + 8(y-4) + 15 = 0$.

In the next example, we notice that $(\sqrt{x})^2 = x$. Also, remember that when we square both sides of an equation, we may introduce extraneous roots. Be sure to check your answers!

EXAMPLE 9.32

Solve: $x - 3\sqrt{x} + 2 = 0$.

Solution

The $\sqrt{x}$ in the middle term, is squared in the first term $(\sqrt{x})^2 = x$. If we let $u = \sqrt{x}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x - 3\sqrt{x} + 2 = 0$
Rewrite the trinomial to prepare for the substitution.	$(\sqrt{x})^2 - 3\sqrt{x} + 2 = 0$
Let $u = \sqrt{x}$ and substitute.	$u^2 - 3u + 2 = 0$
Solve by factoring.	$(u-2)(u-1) = 0$ $u-2=0,\quad u-1=0$ $u=2,\quad u=1$
Replace u with $\sqrt{x}$.	$\sqrt{x}=2,\quad \sqrt{x}=1$
Solve for x, by squaring both sides.	$x=4,\quad x=1$
Check:	

$$\begin{array}{cc} x=4 & x=1 \\ x-3\sqrt{x}+2=0 & x-3\sqrt{x}+2=0 \\ 4-3\sqrt{4}+2\stackrel{?}{=}0 & 1-3\sqrt{1}+2\stackrel{?}{=}0 \\ 4-6+2\stackrel{?}{=}0 & 1-3+2\stackrel{?}{=}0 \\ 0=0\checkmark & 0=0\checkmark \end{array}$$

TRY IT : : 9.63 Solve: $x - 7\sqrt{x} + 12 = 0$.

TRY IT : : 9.64 Solve: $x - 6\sqrt{x} + 8 = 0$.

Substitutions for rational exponents can also help us solve an equation in quadratic form. Think of the properties of exponents as you begin the next example.

EXAMPLE 9.33

Solve: $x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$.

Solution

The $x^{\frac{1}{3}}$ in the middle term is squared in the first term $\left(x^{\frac{1}{3}}\right)^2 = x^{\frac{2}{3}}$. If we let $u = x^{\frac{1}{3}}$ and substitute, our trinomial will be in $ax^2 + bx + c = 0$ form.

	$x^{\frac{2}{3}} - 2x^{\frac{1}{3}} - 24 = 0$
Rewrite the trinomial to prepare for the substitution.	$\left(x^{\frac{1}{3}}\right)^2 - 2\left(x^{\frac{1}{3}}\right) - 24 = 0$
Let $u = x^{\frac{1}{3}}$ and substitute.	$u^2 - 2u - 24 = 0$

Solve by factoring.	$(u-6)(u+4)=0$ $u-6=0, \quad u+4=0$ $u=6, \quad u=-4$
Replace u with $x^{\frac{1}{3}}$.	$x^{\frac{1}{3}}=6, \quad x^{\frac{1}{3}}=-4$
Solve for x by cubing both sides.	$\left(x^{\frac{1}{3}}\right)^3=(6)^3, \quad \left(x^{\frac{1}{3}}\right)^3=(-4)^3$ $x=216, \quad x=-64$

Check:

$x=216$	$x=-64$
$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$	$x^{\frac{2}{3}}-2x^{\frac{1}{3}}-24=0$
$(216)^{\frac{2}{3}}-2(216)^{\frac{1}{3}}-24\stackrel{?}{=}0$	$(-64)^{\frac{2}{3}}-2(-64)^{\frac{1}{3}}-24\stackrel{?}{=}0$
$36-12-24\stackrel{?}{=}0$	$16+8-24\stackrel{?}{=}0$
$0=0$ ✓	$0=0$ ✓

TRY IT : : 9.65 Solve: $x^{\frac{2}{3}}-5x^{\frac{1}{3}}-14=0$.

TRY IT : : 9.66 Solve: $x^{\frac{1}{2}}+8x^{\frac{1}{4}}+15=0$.

In the next example, we need to keep in mind the definition of a negative exponent as well as the properties of exponents.

EXAMPLE 9.34

Solve: $3x^{-2}-7x^{-1}+2=0$.

Solution

The x^{-1} in the middle term is squared in the first term $\left(x^{-1}\right)^2=x^{-2}$. If we let $u=x^{-1}$ and substitute, our trinomial will be in $ax^2+bx+c=0$ form.

	$3x^{-2}-7x^{-1}+2=0$
Rewrite the trinomial to prepare for the substitution.	$3(x^{-1})^2-7(x^{-1})+2=0$
Let $u=x^{-1}$ and substitute.	$3u^2-7u+2=0$
Solve by factoring.	$(3u-1)(u-2)=0$
	$3u-1=0, \quad u-2=0$
	$u=\frac{1}{3}, \quad u=2$
Replace u with x^{-1}.	$x^{-1}=\frac{1}{3}, \quad x^{-1}=2$
Solve for x by taking the reciprocal since $x^{-1}=\frac{1}{x}$.	$x=3, \quad x=\frac{1}{2}$

Check:

$$x = 3 \qquad\qquad x = \frac{1}{2}$$

$$3x^{-2} - 7x^{-1} + 2 = 0 \qquad 3x^{-2} - 7x^{-1} + 2 = 0$$

$$3(3)^{-2} - 7(3)^{-1} + 2 \stackrel{?}{=} 0 \qquad 3\left(\frac{1}{2}\right)^{-2} - 7\left(\frac{1}{2}\right)^{-1} + 2 \stackrel{?}{=} 0$$

$$3\left(\frac{1}{9}\right) - 7\left(\frac{1}{3}\right) + 2 \stackrel{?}{=} 0 \qquad 3(4) - 7(2) + 2 \stackrel{?}{=} 0$$

$$\frac{1}{3} - \left(\frac{7}{3}\right) + \frac{6}{3} \stackrel{?}{=} 0 \qquad 12 - 14 + 2 \stackrel{?}{=} 0$$

$$0 = 0 \checkmark \qquad 0 = 0 \checkmark$$

TRY IT : : 9.67 Solve: $8x^{-2} - 10x^{-1} + 3 = 0.$

TRY IT : : 9.68 Solve: $6x^{-2} - 23x^{-1} + 20 = 0.$

MEDIA : :

Access this online resource for additional instruction and practice with solving quadratic equations.

- **Solving Equations in Quadratic Form (https://openstax.org/l/37QuadForm4)**

9.4 EXERCISES

Practice Makes Perfect

Solve Equations in Quadratic Form

In the following exercises, solve.

155. $x^4 - 7x^2 + 12 = 0$

156. $x^4 - 9x^2 + 18 = 0$

157. $x^4 - 13x^2 - 30 = 0$

158. $x^4 + 5x^2 - 36 = 0$

159. $2x^4 - 5x^2 + 3 = 0$

160. $4x^4 - 5x^2 + 1 = 0$

161. $2x^4 - 7x^2 + 3 = 0$

162. $3x^4 - 14x^2 + 8 = 0$

163. $(x - 3)^2 - 5(x - 3) - 36 = 0$

164. $(x + 2)^2 - 3(x + 2) - 54 = 0$

165. $(3y + 2)^2 + (3y + 2) - 6 = 0$

166. $(5y - 1)^2 + 3(5y - 1) - 28 = 0$

167. $\left(x^2 + 1\right)^2 - 5\left(x^2 + 1\right) + 4 = 0$

168. $\left(x^2 - 4\right)^2 - 4\left(x^2 - 4\right) + 3 = 0$

169. $2\left(x^2 - 5\right)^2 - 5\left(x^2 - 5\right) + 2 = 0$

170. $2\left(x^2 - 5\right)^2 - 7\left(x^2 - 5\right) + 6 = 0$

171. $x - \sqrt{x} - 20 = 0$

172. $x - 8\sqrt{x} + 15 = 0$

173. $x + 6\sqrt{x} - 16 = 0$

174. $x + 4\sqrt{x} - 21 = 0$

175. $6x + \sqrt{x} - 2 = 0$

176. $6x + \sqrt{x} - 1 = 0$

177. $10x - 17\sqrt{x} + 3 = 0$

178. $12x + 5\sqrt{x} - 3 = 0$

179. $x^{\frac{2}{3}} + 9x^{\frac{1}{3}} + 8 = 0$

180. $x^{\frac{2}{3}} - 3x^{\frac{1}{3}} = 28$

181. $x^{\frac{2}{3}} + 4x^{\frac{1}{3}} = 12$

182. $x^{\frac{2}{3}} - 11x^{\frac{1}{3}} + 30 = 0$

183. $6x^{\frac{2}{3}} - x^{\frac{1}{3}} = 12$

184. $3x^{\frac{2}{3}} - 10x^{\frac{1}{3}} = 8$

185. $8x^{\frac{2}{3}} - 43x^{\frac{1}{3}} + 15 = 0$

186. $20x^{\frac{2}{3}} - 23x^{\frac{1}{3}} + 6 = 0$

187. $x + 8x^{\frac{1}{2}} + 7 = 0$

188. $2x - 7x^{\frac{1}{2}} = 15$

189. $6x^{-2} + 13x^{-1} + 5 = 0$

190. $15x^{-2} - 26x^{-1} + 8 = 0$

191. $8x^{-2} - 2x^{-1} - 3 = 0$

192. $15x^{-2} - 4x^{-1} - 4 = 0$

Writing Exercises

193. Explain how to recognize an equation in quadratic form.

194. Explain the procedure for solving an equation in quadratic form.

Self Check

ⓐ *After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.*

I can...	Confidently	With some help	No-I don't get it!
solve equations in quadratic form.			

ⓑ *On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?*