5

Linear First-Order Equations

"Linear" first-order differential equations make up another important class of differential equations that commonly arise in applications and are relatively easy to solve (in theory). As with the notion of 'separability', the idea of 'linearity' for first-order equations can be viewed as a simple generalization of the notion of direct integrability, and a relatively straightforward (though, perhaps, not so intuitively obvious) method will allow us to put any first-order linear equation into a form that can be relatively easily integrated. We will derive this method in a short while (after, of course, describing just what it means for a first-order equation to be "linear").

By the way, the criteria given here for a differential equation being linear will be extended later to higher-order differential equations, and a rather extensive theory will be developed to handle linear differential equations of any order. That theory is not needed here; in fact, it would be of very limited value. And, to be honest, the basic technics we'll develop in this chapter are only of limited use when it comes to solving higher-order linear equations. However, these basic technics involve an "integrating factor", which is something we'll be able to generalize a little bit later (in chapter 7) to help solve much more general first-order differential equations.

5.1 Basic Notions

Definitions

A first-order differential equation is said to be *linear* if and only if it can be written as

$$\frac{dy}{dx} = f(x) - p(x)y \tag{5.1}$$

or, equivalently, as

$$\frac{dy}{dx} + p(x)y = f(x) \tag{5.2}$$

where $p(x)$ and $f(x)$ are known functions of x only.

Equation (5.2) is normally considered to be the "standard" form for first-order linear equations. Note that the only appearance of y in a linear equation (other than in the derivative) is in a term where y alone is multiplied by some formula of x. If there is any other functions of y appearing in the equation after you've isolated the derivative, then the equation is not linear.

!► ***Example 5.1:*** *Consider the differential equation*

$$x\frac{dy}{dx} + 4y - x^3 = 0 \quad .$$

Solving for the derivative, we get

$$\frac{dy}{dx} = \frac{x^3 - 4y}{x} = x^2 - \frac{4}{x}y \quad ,$$

which is

$$\frac{dy}{dx} = f(x) - p(x)y$$

with

$$p(x) = \frac{4}{x} \qquad \text{and} \qquad f(x) = x^2 \quad .$$

Adding $^4/_x \cdot y$ *to both sides, we then get the equation in standard form,*

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 \quad ,$$

On the other hand

$$\frac{dy}{dx} + \frac{4}{x}y^2 = x^2$$

is not linear because of the y^2 *.*

In testing whether a given first-order differential equation is linear, it does not matter whether you attempt to rewrite the equation as

$$\frac{dy}{dx} = f(x) - p(x)y$$

or as

$$\frac{dy}{dx} + p(x)y = f(x) \quad .$$

If you can put it into either form, the equation is linear. You may prefer the first, simply because it is a natural form to look for after solving the equation for the derivative. However, because the second form (the standard form) is more suited for the methods normally used for solving these equations, more experienced workers typically prefer that form.

!► ***Example 5.2:*** *Consider the equation*

$$x^2\frac{dy}{dx} + x^3[y - \sin(x))] = 0 \quad .$$

Dividing through by x^2 *and doing a little multiplication and addition converts the equation to*

$$\frac{dy}{dx} + xy = x\sin(x) \quad ,$$

which is the standard form for a linear equation. So this differential equation is linear.

It is possible for a linear equation

$$\frac{dy}{dx} + p(x)y = f(x)$$

to also be a type of equation we've already studied. For example, if $p(x) = 0$ then the equation is

$$\frac{dy}{dx} = f(x) \quad ,$$

which is directly integrable. If, instead, $f(x) = 0$, the equation can be rewritten as

$$\frac{dy}{dx} = -p(x)y \quad ,$$

showing that it is separable. In addition, you can easily verify that a linear equation is separable if $f(x)$ is any constant multiple of $p(x)$.

If a linear equation is also directly integrable or separable, then it can be solved using methods already discussed. Otherwise, a small trick turns out to be very useful.

Deriving the Trick for Solving

Suppose we want to solve some first-order linear equation

$$\frac{dy}{dx} + py = f \tag{5.3}$$

(for brevity, $p = p(x)$ and $f = f(x)$). To avoid triviality, let's assume $p(x)$ is not always 0. Whether $f(x)$ vanishes or not will not be relevant.

The small trick to solving equation (5.3) comes from the product rule for derivatives: If μ and y are two functions of x, then

$$\frac{d}{dx}[\mu y] = \frac{d\mu}{dx}y + \mu\frac{dy}{dx} \quad .$$

Rearranging the terms on the right side, we get

$$\frac{d}{dx}[\mu y] = \mu\frac{dy}{dx} + \frac{d\mu}{dx}y \quad ,$$

and the right side of this equation looks a little like the left side of equation (5.3). To get a better match, let's multiply equation (5.3) by μ,

$$\mu\frac{dy}{dx} + \mu py = \mu f \quad .$$

With luck, the left side of this equation will match the right side of the last equation for the product rule, and we will have

$$\begin{aligned} \frac{d}{dx}[\mu y] &= \mu\frac{dy}{dx} + \frac{d\mu}{dx}y \\ &= \mu\frac{dy}{dx} + \mu py = \mu f \quad . \end{aligned} \tag{5.4}$$

This, of course, requires that

$$\frac{d\mu}{dx} = \mu p \quad .$$

Assuming this requirement is met, the equations in (5.4) hold. Cutting out the middle of that (and recalling that f and μ are functions of x only), we see that the differential equation reduces to

$$\frac{d}{dx}[\mu y] = \mu(x) f(x) \quad . \tag{5.5}$$

The advantage of having our differential equation in this form is that we can actually integrate both sides with respect to x, with the left side being especially easy since it is just a derivative with respect to x.

The function μ is called an *integrating factor* for the differential equation. As noted in the derivation, it must satisfy

$$\frac{d\mu}{dx} = \mu p \quad . \tag{5.6}$$

This is a simple separable differential equation for μ (remember, $p = p(x)$ is a known function). Any nonzero solution to this can be used as an integrating factor (the zero solution, $\mu = 0$, would simplify matters too much!). Applying the approach we learned for separable differential equations, we divide through by μ, integrate, and solve the resulting equation for μ:

$$\int \frac{1}{\mu}\frac{d\mu}{dx}\,dx = \int p(x)\,dx$$

$$\Longrightarrow \qquad \ln|\mu| = \int p(x)\,dx$$

$$\Longrightarrow \qquad \mu = \pm e^{\int p(x)\,dx}$$

Since we only need one function $\mu(x)$ satisfying requirement (5.6), we can drop both the "$\pm$" and any arbitrary constant arising from the integration of $p(x)$. This leaves us with a relatively simple formula for our integrating factor; namely,

$$\mu(x) = e^{\int p(x)\,dx} \tag{5.7}$$

where it is understood that we can let the constant of integration be zero.

5.2 Solving First-Order Linear Equations

As we just derived, the real 'trick' to solving a first-order linear equation is to reduce it to an easily integrated form via the use of an integrating factor. Let me outline a procedure for actually carrying out the necessary steps. To illustrate these steps, we will immediately use them to find the general solution to the equation from example 5.1,

$$x\frac{dy}{dx} + 4y = x^3 \quad .$$

The Procedure:

1. Get the equation into the standard form for first-order linear differential equations,

$$\frac{dy}{dx} + p(x)y = f(x) \quad .$$

For our example, we just divide through by x, obtaining

$$\frac{dy}{dx} + \frac{4}{x}y = x^2 \quad .$$

As noted in example 5.1, this is the desired form with

$$p(x) = \frac{4}{x} \qquad \text{and} \qquad f(x) = x^2 \quad .$$

2. Compute an integrating factor

$$\mu(x) = e^{\int p(x)\,dx} \quad .$$

Remember, since we only need one integrating factor, we can let the constant of integration be zero here.

For our example

$$\mu(x) = e^{\int p(x)\,dx} = e^{\int \frac{4}{x}\,dx} = e^{4\ln|x|} = x^4 \quad .$$

3a. Multiply the differential equation (in standard form) by the integrating factor,

$$\mu\left[\frac{dy}{dx} + p(x)y = f(x)\right]$$

$$\Longrightarrow \qquad \mu\frac{dy}{dx} + \mu py = \mu f \quad ,$$

b. and observe that, via the product rule and choice of μ, the left side can be written as the derivative of the product of μ and y,

$$\underbrace{\mu\frac{dy}{dx} + \mu py}_{\frac{d}{dx}[\mu y]} = \mu f \quad ,$$

c. and then rewrite the differential equation as

$$\frac{d}{dx}[\mu y] = \mu f \quad ,$$

For our example, $\mu = x^4$. Multiplying our equation by this and proceeding through the three substeps above, yields

$$x^4\left[\frac{dy}{dx} + \frac{4}{x}y = x^2\right]$$

$$\Longrightarrow \qquad \underbrace{x^4\frac{dy}{dx} + 4x^3y}_{\frac{d}{dx}[x^4 y]} = x^6$$

$$\Longrightarrow \qquad \frac{d}{dx}[x^4 y] = x^6 \quad .$$

4. Integrate with respect to x both sides of the last equation obtained,

$$\int \frac{d}{dx}[\mu y]\,dx = \int \mu(x)\,f(x)\,dx$$

$$\Longrightarrow \qquad \mu y = \int \mu(x)\,f(x)\,dx \quad .$$

Don't forget the arbitrary constant here!

Integrating the last equation in our example,

$$\int \frac{d}{dx}[x^4 y]\,dx = \int x^6\,dx$$

$$\Longrightarrow \qquad x^4 y = \frac{1}{7}x^7 + c \quad .$$

5. Finally, solve for y by dividing through by μ.

For our example,

$$y = x^{-4}\left[\frac{1}{7}x^7 + c\right] = \frac{1}{7}x^3 + cx^{-4} \quad .$$

It is possible to use the above procedure to derive a explicit formula for computing y from $p(x)$ and $f(x)$. Unfortunately, it is not a particularly simple formula, and those who attempt to memorize it typically make more mistakes than those who simply remember the above procedure. So I won't tell you that formula, yet.[1]

!►*Example 5.3:* *Consider*

$$e^x \frac{dy}{dx} = 20 + 3e^x y \quad , \quad y(0) = 7$$

Subtracting $3e^x y$ *from both sides and then multiplying through by* e^{-x} *puts this linear differential equation into the desired form,*

$$\frac{dy}{dx} - 3y = 20e^{-x} \quad .$$

So $p(x) = -3$*, and our integrating factor is*

$$\mu = \mu(x) = e^{\int -3\,dx} = e^{-3x} \quad .$$

Multiplying the differential equation by μ *and following the rest of the steps in our procedure gives us the following:*

$$e^{-3x}\left[\frac{dy}{dx} - 3y = 20e^{-x}\right]$$

$$\Longrightarrow \qquad \underbrace{e^{-3x}\frac{dy}{dx} - 3e^{-3x}y}_{\frac{d}{dx}[e^{-3x}y]} = 20e^{-4x}$$

[1] If you must see this formula, glance ahead to theorem 5.1 on page 113.

$$\begin{aligned}
\Longrightarrow \qquad \frac{d}{dx}[e^{-3x}y] &= 20e^{-4x} \\
\Longrightarrow \qquad \int \frac{d}{dx}[e^{-3x}y]\,dx &= \int 20e^{-4x}\,dx \\
\Longrightarrow \qquad e^{-3x}y &= -5e^{-4x} + c \\
\Longrightarrow \qquad y &= e^{3x}\left[-5e^{-4x} + c\right] \quad .
\end{aligned}$$

So the general solution to our differential equation is

$$y(x) = -5e^{-x} + ce^{3x} \quad .$$

Using this formula for $y(x)$ *with the initial condition gives us*

$$7 = y(0) = -5e^{-0} + ce^{3\cdot 0} = -5 + c$$

Thus,

$$c = 7 + 5 = 12 \quad ,$$

and the solution to the given initial-value problem is

$$y(x) = -5e^{-x} + 12e^{3x} \quad .$$

Let us briefly get back to our requirement for $\mu = \mu(x)$ being an integrating factor for

$$\frac{dy}{dx} + py = f \quad .$$

That requirement was equation (5.6),

$$\frac{d\mu}{dx} = \mu p \quad .$$

Now, in computing this μ, you will often get something like

$$\mu(x) = |\mu_0(x)|$$

where $\mu_0(x)$ is a relatively simple continuous function (e.g., $\mu(x) = |\sin(x)|$). Consequently, on any interval over which the graph of $\mu_0(x)$ never crosses the X–axis,

$$\mu_0(x) = \mu(x) \qquad \text{or} \qquad \mu_0(x) = -\mu(x) \quad .$$

Either way,

$$\frac{d\mu_0}{dx} = \frac{d[\pm\mu]}{dx} = \pm\frac{d\mu}{dx} = \pm\mu p = \mu_0 p \quad .$$

So μ_0 also satisfies the requirement for being an integrating factor for the given differential equation. This means that, if in computing μ you do get something like

$$\mu(x) = |\mu_0(x)|$$

where $\mu_0(x)$ is a relatively simple function, then you can ignore the absolute value brackets and just use μ_0 for your integrating factor.

!►Example 5.4: *Consider solving the linear differential equation*

$$\frac{dy}{dx} + \cot(x)y = x\csc(x) \quad .$$

This equation is already in the desired form. In a case like this, it is often a good idea to see what the equation looks like in terms of sines and cosines,

$$\frac{dy}{dx} + \left[\frac{\cos(x)}{\sin(x)}\right]y = \frac{x}{\sin(x)} \quad .$$

To find $\mu = e^{\int p\,dx}$ *, first observe that, ignoring the constant of integration,*

$$\int p(x)\,dx = \int \frac{\cos(x)}{\sin(x)}\,dx = \int \frac{\frac{d}{dx}\sin(x)}{\sin(x)}\,dx = \ln|\sin(x)| \quad .$$

So

$$\mu = \mu(x) = e^{\int p(x)\,dx} = e^{\ln|\sin(x)|} = |\sin(x)| \quad .$$

As discussed above, we can just drop the $|\cdot|$ *and use* $\sin(x)$ *for the integrating factor. Doing so, and stepping through the rest of our procedure, we have*

$$\sin(x)\left[\frac{dy}{dx} + \frac{\cos(x)}{\sin(x)}y = \frac{x}{\sin(x)}\right]$$

$$\Longrightarrow \quad \underbrace{\sin(x)\frac{dy}{dx} + \cos(x)y}_{\frac{d}{dx}[\sin(x)y]} = x$$

$$\Longrightarrow \quad \int \frac{d}{dx}[\sin(x)y]\,dx = \int x\,dx$$

$$\Longrightarrow \quad \sin(x)y = \frac{1}{2}x^2 + c_1$$

$$\Longrightarrow \quad y = \frac{x^2 + c}{2\sin(x)} \quad .$$

5.3 On Using Definite Integrals with Linear Equations

Integration arises twice in our method for solving

$$\frac{dy}{dx} + p(x)y = f(x) \quad .$$

It first arises when we integrate p to get the integrating factor,

$$\mu(x) = e^{\int p(x)\,dx} \quad .$$

It then is needed again when we then integrate both sides of the corresponding equation

$$\frac{d}{dx}[\mu y] = \mu f \quad .$$

At either point, of course, we could use definite integrals instead of indefinite integrals.

Let's first look at what happens when we integrate both sides of the last equation using definite integrals. Remember, everything is a function of x , so this equation can be written a bit more explicitly as

$$\frac{d}{dx}[\mu(x)y(x)] = \mu(x)f(x) \quad .$$

As before, to avoid having x represent two different entities, we replace the x's with another variable, say, s , and rewrite our current differential equation as

$$\frac{d}{ds}[\mu(s)y(s)] = \mu(s)f(s) \quad .$$

Then we pick a convenient lower limit a for our integration and integrate each side of the above with respect to s from $s = a$ to $s = x$,

$$\int_a^x \frac{d}{ds}[\mu(s)y(s)]\,ds = \int_a^x \mu(s)f(s)\,ds \quad . \tag{5.8}$$

But

$$\int_a^x \frac{d}{ds}[\mu(s)y(s)]\,ds = \mu(s)y(s)\Big|_a^x = \mu(x)y(x) - \mu(a)y(a) \quad .$$

So equation (5.8) reduces to

$$\mu(x)y(x) - \mu(a)y(a) = \int_a^x \mu(s)f(s)\,ds \quad ,$$

Solving this for $y(x)$ yields

$$y(x) = \frac{1}{\mu(x)}\left[\mu(a)y(a) + \int_a^x \mu(s)f(s)\,ds\right] \quad . \tag{5.9}$$

This is not a simple enough formula to be worth memorizing (especially since you still have to remember what μ is). Nonetheless, it is a formula worth knowing about for at least two good reasons:

1. This formula can automatically take into account an initial value $y(x_0) = y_0$. All we have to do is to choose the lower limit a to be x_0 . Then formula (5.9) tells us that the solution to

$$\frac{dy}{dx} + py = f \qquad \text{with} \quad y(x_0) = y_0$$

   is

$$y = \frac{1}{\mu(x)}\left[\mu(x_0)y_0 + \int_{x_0}^x \mu(s)f(s)\,ds\right] \quad .$$

2. Even if we cannot determine a relatively nice formula for integral of μf (for a given choice of μ and f), the value of the integral in formula (5.9) can, in practice, still be accurately computed for desired values of x using numerical integration routines found in standard computer math packages. Indeed, using any of these packages and formula (5.9), you could probably program a computer to accurately compute $y(x)$ for a number of values of x and use these values to produce a very accurate graph of y .

!►Example 5.5: *Consider solving*

$$\frac{dy}{dx} - 2xy = 2 \qquad \text{with} \quad y(0) = 1 \quad .$$

The differential equation is clearly linear and in the desired form for the first step of our procedure. Computing the integrating factor, we find that, here,

$$\mu = e^{\int p(x)\,dx} = e^{\int [-2x]\,dx} = e^{-x^2+c} \quad .$$

Choosing, as we may, c to be zero, we then get

$$\mu(x) = e^{-x^2} \quad .$$

Plugging this into formula (5.9) (and choosing $a = 0$ since we have $y(0) = 1$ as the initial condition) yields

$$\begin{aligned} y(x) &= \frac{1}{\mu(x)}\left[\mu(0)y(0) + \int_0^x \mu(s)f(s)\,ds\right] \\ &= \frac{1}{e^{-x^2}}\left[e^{-0^2}\cdot 1 + \int_0^x e^{-s^2}2\,ds\right] \\ &= e^{x^2}\left[1 + 2\int_0^x e^{-s^2}\,ds\right] \quad . \end{aligned}$$

This is the solution to our initial-value problem. The integral,

$$\int_0^x e^{-s^2}\,ds \quad ,$$

was left unevaluated because no one has yet found a "nice" formula for this integral. At best, we can 'hide' this integral by using the error function (see page 30), rewriting our formula for y as

$$y(x) = e^{x^2}\left[1 + \sqrt{\pi}\,\mathrm{erf}(x)\right] \quad .$$

Still, to find the value of, say, $y(4)$, we would have to either numerically approximate the integral in

$$y(4) = e^{4^2}\left[1 + 2\int_0^4 e^{-s^2}\,ds\right]$$

or look up the value of the error function in

$$y(4) = e^{4^2}\left[1 + \sqrt{\pi}\,\mathrm{erf}(4)\right] \quad .$$

Either way, a decent computer math package could be helpful.

As already noted, we could also use a definite integral in determining the integrating factor. This means μ would be given by

$$\mu(x) = e^{\int_a^x p(s)\,ds}$$

where a was any appropriate lower limit. Naturally, if we had an initial condition $y(x_0) = y_0$, it would make sense to let $a = x_0$ (though this is less important here than it was in formula (5.9)). In practice, there is little to be gained in using a definite integral in the computation of μ *unless* there is not a reasonable formula for the integral of p. Then we are pretty well forced into using a definite integral to compute $\mu(x)$ and to computing this integral numerically for each value of x of interest. That, in turn, would pretty well force us to compute $y(x)$ for each x of interest by using numerical computation of formula (5.9).

5.4 Integrability and the Existence and Uniqueness of Solutions

If you check, you will see that our derivation of the definite integral formula

$$y(x) \;=\; \frac{1}{\mu(x)}\left[\mu(x_0)y_0 \;+\; \int_{x_0}^{x} \mu(s)\,f(s)\,ds\right] \qquad \text{with} \quad \mu(x) \;=\; \exp\left(\int_{x_0}^{x} p(s)\,ds\right)$$

as a solution to the initial-value problem

$$\frac{dy}{dx} \;+\; p(x)y \;=\; f(x) \qquad \text{with} \quad y(x_0) = y_0$$

merely required that y be one solution to this problem, and that p and f be 'sufficiently integrable' for the existence of the integrals involving them. Conversely, as long as p and f are 'sufficiently integrable', you can use elementary calculus to differentiate the above definite integral formula and verify that the y defined by this formula is a solution to the above initial-value problem (see problem 5.5). Thus, the above definite integral formula gives us *the one and only* solution to the above initial-value problem, provided p and f are 'sufficiently integrable'.

Just what is 'sufficiently integrable'? Basically, we want the integrals

$$\int_{x_0}^{x} p(s)\,ds \qquad \text{and} \qquad \int_{x_0}^{x} \mu(s)\,f(s)\,ds$$

to be well-defined, continuous functions of x in whatever interval of interest (α, β) we have. (Note that this ensures

$$\mu(x) \;=\; \exp\left(\int_{x_0}^{x} p(s)\,ds\right)$$

is never zero in this interval.) Certainly, p and f will be 'sufficiently integrable' if they are continuous on (α, β). But continuity is not necessary; p and f can have a few discontinuities provided these discontinuities are not too bad. In particular, we can allow the same piecewise-defined functions considered back in section 2.4. That (along with theorem 2.1 on page 35) gives us the following existence and uniqueness theorem for initial-value problems involving first-order linear differential equations.

Theorem 5.1 (existence and uniqueness)
Let p and f be functions that are continuous except for, at most, a finite number of finite-jump discontinuities in an interval (α, β). Also let x_0 and y_0 be any two numbers with $\alpha < x_0 < \beta$. Then the initial-value problem

$$\frac{dy}{dx} \;+\; p(x)y \;=\; f(x) \qquad \textit{with} \quad y(x_0) = y_0$$

has exactly one solution over the interval (α, β), and that solution is given by

$$y(x) \;=\; \frac{1}{\mu(x)}\left[\mu(x_0)y_0 \;+\; \int_{x_0}^{x} \mu(s)\,f(s)\,ds\right] \qquad \textit{with} \quad \mu(x) \;=\; \exp\left(\int_{x_0}^{x} p(s)\,ds\right) \quad .$$

Additional Exercises

5.1. *Determine whether each of the following differential equations is or is not linear, and, if it is linear, rewrite the equation in standard form,*

$$\frac{dy}{dx} + p(x)y = f(x) \quad .$$

a. $x^2\frac{dy}{dx} + 3x^2y = \sin(x)$

b. $y^2\frac{dy}{dx} + 3x^2y = \sin(x)$

c. $\frac{dy}{dx} - xy^2 = \sqrt{x}$

d. $\frac{dy}{dx} = 1 + (xy+3y)^2$

e. $\frac{dy}{dx} = 1 + xy + 3y$

f. $\frac{dy}{dx} = 4y + 8$

g. $\frac{dy}{dx} - e^{2x} = 0$

h. $\frac{dy}{dx} = \sin(x)\,y$

i. $\frac{dy}{dx} + 4y = y^3$

j. $x\frac{dy}{dx} + \cos\left(x^2\right) = 827y$

5.2. *Using the methods developed in this chapter, find the general solution to each of the following first-order linear differential equations:*

a. $\frac{dy}{dx} + 2y = 6$

b. $\frac{dy}{dx} + 2y = 20e^{3x}$

c. $\frac{dy}{dx} = 4y + 16x$

d. $\frac{dy}{dx} - 2xy = x$

e. $x\frac{dy}{dx} + 3y - 10x^2 = 0$

f. $x^2\frac{dy}{dx} + 2xy = \sin(x)$

g. $x\frac{dy}{dx} = \sqrt{x} + 3y$

h. $\cos(x)\frac{dy}{dx} + \sin(x)\,y = \cos^2(x)$

i. $x\frac{dy}{dx} + (5x+2)y = \frac{20}{x}$

j. $2\sqrt{x}\frac{dy}{dx} + y = 2xe^{-\sqrt{x}}$

5.3. *Find the solution to each of the following initial-value problems using the methods developed in this chapter:*

a. $\frac{dy}{dx} - 3y = 6 \quad$ *with* $\quad y(0) = 5$

b. $\frac{dy}{dx} - 3y = 6 \quad$ *with* $\quad y(0) = -2$

c. $\frac{dy}{dx} + 5y = e^{-3x} \quad$ *with* $\quad y(0) = 0$

d. $2\frac{dy}{dx} + 3y = 20x^2 \quad$ *with* $\quad y(1) = 10$

e. $x\frac{dy}{dx} = y + x^2\cos(x) \quad$ *with* $\quad y\left(\frac{\pi}{2}\right) = 0$

f. $(1+x^2)\frac{dy}{dx} = x\left[3+3x^2-y\right] \quad$ *with* $\quad y(2) = 8$

5.4. *Express the answer to each of the following initial-value problems in terms of definite integrals:*

a. $\dfrac{dy}{dx} + 6xy = \sin(x)$ *with* $y(0) = 4$

b. $x^2\dfrac{dy}{dx} + xy = \sqrt{x}\sin(x)$ *with* $y(2) = 5$

c. $x\dfrac{dy}{dx} - y = x^2 e^{-x^2}$ *with* $y(3) = 8$

5.5. *Let* (α, β) *be an interval, and let* x_0 *and* y_0 *be any two numbers with* $\alpha < x_0 < \beta$ *. Assume* p *and* f *are functions continuous at all but, at most, a finite number of points in* (α, β) *, and that each of these discontinuities is a finite-jump discontinuity. Define* $\mu(x)$ *and* $y(x)$ *by*

$$\mu(x) = \exp\left(\int_{x_0}^{x} p(s)\, ds\right) \quad .$$

and

$$y(x) = \frac{1}{\mu(x)}\left[\mu(x_0)y_0 + \int_{x_0}^{x} \mu(s) f(s)\, ds\right]$$

a. *Compute the first derivatives of* μ *and* y *.*

b. *Verify that* y *satisfies the initial condition* $y(x_0) = y_0$ *as well as the differential equation*

$$\frac{dy}{dx} + p(x)y = f(x)$$

on (α, β) *.*